

CMPT260.3-01 Final Exam

December 9, 2006

Three (3) hours.

Closed book. No calculators or computers.

Instructions: Answer all questions in a University of Saskatchewan examination booklet. A portion of the marks awarded will be for the style and clarity of your answer. Do not use a hard pencil. There are 100 marks on this exam, the values appear in bold next to the questions.

If you are a student who has been excused from an assignment/quiz/midterm during the term, please write a note to this effect on the cover of your exam booklet.

1. Logic

- a. **(4)** Give truth tables for $P \vee Q$ and $P \Rightarrow Q$.
- b. **(4)** Each of four cards in a deck has a single character printed on each side. A card dealer tells you the following about this deck. "If a card has a vowel on one side, then it has an even number on the other side". The four cards are dealt out and you see the following characters:

A B 8 5

How many cards (and which ones) must you turn over to verify the truth of the dealer's statement?

- c. **(6)** Give the truth table for

$$(C \vee \neg B) \Leftrightarrow (B \Rightarrow (B \wedge C)).$$

- d. **(4)** Prove

$$(A \Leftrightarrow (A \wedge B)) \vdash A \Rightarrow B$$

- e. **(4)** Prove

$$A \Rightarrow C, B \Rightarrow \neg C, A \vdash \neg B$$

- f. **(4)** Simplify the following expression by moving all negations in:

$$\neg(P \vee (Q \wedge \neg P) \vee \neg(\neg Q \wedge P)).$$

2. Induction

- a) **(4)** Give the definition of addition.
- b) **(6)** Using only the Peano postulates, and the definition of addition, prove the associative property of addition:

$$(a + b) + n = a + (b + n).$$

- c) **(6)** The *height* of a single node is 0. The height of a binary tree is the maximum of the heights of its two subtrees, plus 1. Show, by induction, that the number of nodes in a complete (no nodes missing) binary tree of height n is $2^{n+1} - 1$.

4
6

DO EITHER QUESTION 3 OR QUESTION 4.

3. (10) More induction

- a. Define exponentiation.
- b. Assume that the following two theorems have been proven:

Theorem 9: $(ab)^x = a^x b^x$.

Theorem 10: $a^n a^m = a^{n+m}$.

Using only these two theorems about exponentiation, and familiar facts about addition and multiplication, prove

$$(a^n)^m = a^{nm}.$$

4. (10) Prolog

Define predicates `rotateL(L1, L2)`, and `rotateR(L1, L2)` such that `L2` is the result of rotating all elements of a list `L1` one unit to the left or to the right, respectively. For example:

```
?- rotateL( [], L)
L = []
?- rotateL( [a], L).
L = [a]
?- rotateL( [a,b,c], L).
L = [b,c,a]
?- rotateR( [a,b,c,d], L).
L = [d,a,b,c]
```

Your solution should not call any other predicates, and should return no other answers on backtracking.

5. Functions

- a. **(4)** Show that the function $F: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 - 8$ is a bijection, using the definitions of 1-1 and onto.
- b. **(6)** Show that the function $F: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2 - 8$ is not a bijection.
- c. **(6)** For any natural number k , find a bijection between the natural numbers $(0, 1, 2, 3 \dots)$ and the set $(k, k+1, k+2, k+3, \dots)$. Prove your answer.

6. Sets and Relations

- a. (6) What is the powerset of the set $A = \{a, b, c\}$? If $B = \{0, 1\}$, what is $A \times B$?
- b. (8) Let $S = \{r, s, t, u\}$. Let $R = \{(s, t), (u, r)\}$. Find
 - i. The reflexive closure of R ,
 - ii. The symmetric closure of R ,
 - iii. The transitive closure of R ,
 - iv. The reflexive, symmetric, and transitive closure of R .
- c. (4) Define *equivalence relation*.
- d. (6) Define *set equality*. Prove that set equality is an equivalence relation.

7. (8)

- a. Define what is meant by the *natural join* of two relations.
- b. Let A and B be the relations given by the following tables:

A	Character	Handle	MembershipDate
	Elmer	123	25-04-92
	Bugs	118	19-06-94
	Tweety	617	11-08-96
	Sylvester	622	08-09-96
	Pepe	417	02-11-94

B	Handle	Rank
	417	top
	622	second
	123	second

Give

- a) The restriction of A to persons who joined *strictly before* 1996. (The last two digits of the *MembershipDate* gives the date of joining.) Call this new relation C .
- b) The projection of C onto attributes *Character* and *Handle*. Call the new relation D .
- c) The natural join of D and B .

End of Exam

Equivalences, Propositional Calculus

Law	Name
$P \vee \neg P \equiv T$ $P \wedge \neg P \equiv F$	Excluded middle law Contradiction law
$P \vee F \equiv P$ $P \wedge T \equiv P$	Identity laws
$P \vee T \equiv T$ $P \wedge F \equiv F$	Domination laws
$P \vee P \equiv P$ $P \wedge P \equiv P$	Idempotent laws
$\neg(\neg P) \equiv P$	Double negation law
$P \vee Q \equiv Q \vee P$ $P \wedge Q \equiv Q \wedge P$	Commutative laws
$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$ $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	Associative laws
$(P \vee Q) \wedge (P \vee R) \equiv P \vee (Q \wedge R)$ $(P \wedge Q) \vee (P \wedge R) \equiv P \wedge (Q \vee R)$	Distributive laws
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	De Morgan's laws

Main rules of inference

$A, B \models A \wedge B$	Law of Combination
$A \wedge B \models B$	Law of Simplification
$A \wedge B \models A$	Variant of Law of Simplification
$A \models A \vee B$	Law of Addition
$B \models A \vee B$	Variant of Law of Addition
$A, A \Rightarrow B \models B$	Modus Ponens
$\neg B, A \Rightarrow B \models \neg A$	Modus Tollens
$A \Rightarrow B, B \Rightarrow C \models A \Rightarrow C$	Hypothetical Syllogism
$A \vee B, \neg A \models B$	Disjunctive Syllogism
$A \vee B, \neg B \models A$	Variant of Disjunctive Syllogism
$A \Rightarrow B, \neg A \Rightarrow B \models B$	Law of Cases
$A \Leftrightarrow B \models A \Rightarrow B$	Equivalence Elimination
$A \Leftrightarrow B \models B \Rightarrow A$	Variant of Equivalence Elimination
$A \Rightarrow B, B \Rightarrow A \models A \Leftrightarrow B$	Equivalence Introduction
$A, \neg A \models B$	Inconsistency Law
$B \Rightarrow A \wedge \neg A \models \neg B$	Indirect Proof

~~$P \Leftrightarrow Q$~~

$$P \rightarrow Q = \neg P \vee Q$$

$$P \Leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$$

$$P \Leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$$